

Near Horizon Analysis of Extremal AdS_5 Black Holes

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ABSTRACT: We study the near horizon geometry of extremal black holes in five dimensional gauged supergravity using Sen's entropy function formalism. Special attention is paid to the large black hole limit where the near horizon solution exhibits a universal dependence on the rotation. The physical properties of the large black hole solution are shown to agree with predictions from fluid mechanical description of the dual conformal field theory.

KEYWORDS: [AdS/CFT](#), [black hole entropy](#).

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1. Introduction

The AdS/CFT correspondence [1] is a concrete realization of the holographic principle [2, 3, 4] in the sense that the CFT defined on the boundary of the AdS space is believed to capture the full dynamics of string theory in the bulk. One may say that the boundary of the AdS serves as the “holographic screen.”

Given that the holographic principle, including the notion of the holographic screen, has its origin in black hole thermodynamics, the recurring appearance of black holes in the development of the AdS/CFT correspondence should come as no surprise. For example, recent applications of the AdS/CFT methods to the hydrodynamic regime of strongly

coupled field theories [5, 6, 7, 8, 9, 10] bear strong resemblance to the membrane paradigm of black hole physics [11, 12, 13].

When the AdS space contain a black hole inside, one encounters an interesting situation where the two holographic screens – the black hole horizon and the AdS boundary – exist at the same time. It appears that questions on black hole dynamics can be addressed from the two “dual” points of view. For instance, while traditional approaches to understand the black hole entropy have often considered degrees of freedom living on the horizon, but AdS/CFT suggests that the dual CFT on the AdS boundary should give a microscopic explanation of the entropy.

If both the traditional picture and the AdS/CFT picture make sense, it is conceivable that there exists some mapping which relate physical quantities on the two holographic screens as well as the equations governing them. Exploring the possibility of such a mapping was a key motivation which initiated this work.

Recently, in [14], the description of the CFT in terms of fluid dynamics was used to make striking predictions on the thermodynamics of black holes in AdS. Fluid dynamics is valid when the fluctuation of the CFT is macroscopic. This translates to the condition that the black hole should be “large” in a technical sense we will review in section 2. Among other things, the result of [14] shows that large black holes exhibit universal dependence on the rotation parameters.

The main goal of this paper is to study the properties of large black holes from the opposite side, namely, the black hole horizon, bearing in mind the original motivation mentioned above. For simplicity, we focus on the extremal limit where the entropy function formalism [15, 16, 17] enables us to extract physical informations without dealing with all the field equations. Although most of our analysis does not rely on supersymmetry or number of dimensions, we focus on black holes in gauged supergravity in five dimension because $\text{AdS}_5/\text{CFT}_4$ with supersymmetry offer more examples of dual pairs than any other cases, of which both sides of the duality have explicitly known Lagrangian descriptions.

We derive the near horizon equations of motion and the entropy function for general extremal black holes in AdS_5 . Then we specialize to the large black hole limit and verify that, to the leading order, the near horizon equations admit a universal solution with a factorized dependence on the rotation parameters. We then make a detailed comparison with the predictions from fluid dynamics of [14] and find perfect agreement. In addition, we show how the near horizon equations determine the charge dependence of the thermodynamic potentials (at zero temperature), which cannot be inferred from the analysis of fluid dynamics alone.

This paper is organized as follows. Section 2 covers some preliminary materials. We first establish our notations on gauged supergravity. Then we briefly review the predictions from fluid dynamics [14] on the properties of large black holes in AdS. We also illustrate salient features of the large black hole limit using a simple example in the minimal supergravity. Section 3 and 4 present the main computations of this paper. In section 5, we elaborate on the comparison between our results and the results from fluid dynamics, comment on the implications of our results on the thermodynamic property of the CFT, and conclude with some future directions.

2. Preliminaries

2.1 Gauged supergravity in five dimension

We will work with $D = 5$, $\mathcal{N} = 1$ gauged supergravity theories with n abelian vector fields. The bosonic part of the Lagrangian is

$$(16\pi G)\mathcal{L} = \star(R - 2V) - \frac{1}{2}g_{ij}d\varphi^i \wedge \star d\varphi^j - \frac{1}{2}Q_{IJ}F^I \wedge \star F^J - \frac{1}{6}C_{IJK}F^I \wedge F^J \wedge A^K. \quad (2.1)$$

The $(n - 1)$ scalars, φ^i , parameterize a hypersurface in \mathbb{R}^n ,

$$\frac{1}{6}C_{IJK}X^IX^JX^K = 1. \quad (2.2)$$

The real constant coefficients C_{IJK} define the “real special geometry,”¹

$$X_I \equiv \frac{1}{2}C_{IJK}X^JX^K, \quad Q_{IJ} \equiv X_IX_J - C_{IJK}X^K, \quad g_{ij} \equiv Q_{IJ}\partial_iX^I\partial_jX^J. \quad (2.3)$$

The scalar potential in (2.1) is specified by some constants \bar{X}_I :

$$V = Q^{IJ}\bar{X}_I\bar{X}_J - (X^I\bar{X}_I)^2. \quad (2.4)$$

The supersymmetric extremum of the potential is located at $X_I|_* = \bar{X}_I$, where $V|_* = -6$. This amounts to setting the AdS radius to be unity: $R_{\mu\nu} = -4g_{\mu\nu}$.

All supergravity theories of the form (2.1) contain the minimal supergravity as a closed subsector, as one can see by setting

$$X_I = \bar{X}_I, \quad A^I = \bar{X}^IA. \quad (2.5)$$

Classically, the overall normalization of C_{IJK} is a matter of convention, since the Lagrangian (2.1) is invariant under

$$C_{IJK} \rightarrow \lambda^3 C_{IJK}, \quad A^I \rightarrow \lambda^{-1}A^I, \quad X^I \rightarrow \lambda^{-1}X^I. \quad (2.6)$$

There are many examples of gauged supergravity theories of which the dual CFT is known explicitly. Upon truncation to the massless abelian sector, the famous $\text{AdS}_5 \times S^5$ string theory leads to the $U(1)^3$ theory with $C_{123} = 1$ and all other components of C_{IJK} vanishing. Orbifolds of S^5 corresponds to quiver gauge theories. Another very large class of $\mathcal{N} = 1$ CFTs arise from D3-branes probing toric Calabi-Yau cones and are efficiently described by the brane tiling model [19, 20, 21]. The coefficients C_{IJK} of the corresponding supergravity theories are given by the area of the triangles in the toric diagram [22, 23].

2.2 Predictions from fluid dynamics

According to the AdS/CFT correspondence, a black hole in a global AdS space corresponds to a thermal ensemble of the states in the dual CFT with the same quantum numbers as

¹We follow the conventions of [18] except that $Q_{IJ}^{\text{here}} = 2Q_{IJ}^{\text{there}}$ and $X_I^{\text{here}} = 3X_I^{\text{there}}$.

the black hole. In the $\text{AdS}_5/\text{CFT}_4$ case under discussion, the relevant quantum numbers are the energy E , the charges Q_I and the two angular momenta J_a .

At sufficiently high temperature and/or density, the CFT is expected to admit an effective description in terms of fluid mechanics. In [14], this expectation was combined with known properties of static black holes to make predictions on rotating black holes, which were then verified in all known rotating black hole solutions in the literature. We briefly summarize the result of [14] here.

In the static case, conformal invariance and extensivity dictates that the grand canonical partition function of the fluid take the form

$$\ln Z_{\text{gc}} = VT^3 h(\mu/T), \quad (2.7)$$

where μ^I are the chemical potentials conjugate to the charges Q_I . V and T represent the volume and the overall temperature of the fluid.

It is not known how to compute the function $h(\mu/T)$ or the equation of state of the static fluid directly from the CFT. But, if a charged static black hole solution is known, they can be read off using AdS/CFT . The equation of state of the static fluid is taken as an input into the relativistic Navier-Stokes equations that govern the dynamics of the conformal fluid in general.

A key observation of [14] is that there exists a unique family of ideal fluid solutions to the Navier-Stokes equation in one to one correspondence with rotating black holes, which are simple enough to be written down explicitly. In the five dimensional case, the solution can be summarized by the grand canonical partition function for the rotating fluid,

$$\ln Z_{\text{gc}} = \ln \text{Tr} \exp \left[-\frac{(E - \mu^I Q_I - \Omega^a J_a)}{T} \right] = \frac{VT^3 h(\mu/T)}{(1 - \Omega_1^2)(1 - \Omega_2^2)}, \quad (2.8)$$

where E and Ω^a represent the energy and the angular velocities of the fluid respectively. Note that the solution is universal in the sense that the rotation dependence factorizes and is independent of the function $h(\mu/T)$. All the physical observables, such as the energy, entropy, charges and angular momenta can be obtained by differentiating (2.8) by the conjugate variables.

Fluid dynamics becomes a good description of the CFT if and only if the “mean free path” l_{mfp} of the conformal fluid is much smaller compared to the volume of the fluid which can be taken to be of order one. An estimate of l_{mfp} is given by [14]

$$l_{\text{mfp}} \sim \left. \frac{S}{4\pi E} \right|_{\Omega=0}, \quad (2.9)$$

where S is the entropy of the black hole. For uncharged black holes, l_{mfp} is simply proportional to $1/T$. For charged black holes, l_{mfp} depends also on the chemical potentials such that it is possible to take an extremal ($T \rightarrow 0$) limit while keeping all physical quantities finite. In the following sections, we will show how the results of [14] is reflected on the near horizon geometry of the extremal black holes.

2.3 Extremal black hole in minimal supergravity

In the minimal gauged supergravity, the most general four parameter family of black hole solutions was obtained in [24]. We study the extremal limit of the general solution and take a close look at the near horizon geometry, so that we can use the result of this section as a guide when we study more general theories in later sections.

The solution of [24], with our normalization for A as in (2.5), is given by

$$ds^2 = -\frac{\Delta_\theta [(1 + g^2 r^2) \rho^2 dt + 2q\nu] dt}{\Xi_a \Xi_b \rho^2} + \frac{f}{\rho^4} \left(\frac{\Delta_\theta dt}{\Xi_a \Xi_b} - \omega \right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{2q\nu\omega}{\rho^2} + \frac{r^2 + a^2}{\Xi_a} \sin^2 \theta d\phi_1^2 + \frac{r^2 + b^2}{\Xi_b} \cos^2 \theta d\phi_2^2, \quad (2.10)$$

$$A = \frac{q}{\rho^2} \left(\frac{\Delta_\theta dt}{\Xi_a \Xi_b} - \omega \right), \quad (2.11)$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Xi_a = 1 - a^2, \quad \Xi_b = 1 - b^2, \\ \nu &= b \sin^2 \theta d\phi_1 + a \cos^2 \theta d\phi_2, \quad \omega = a \sin^2 \theta \frac{d\phi_1}{\Xi_a} + b \cos^2 \theta \frac{d\phi_2}{\Xi_b}, \\ \Delta_\theta &= 1 - a^2 \cos^2 \theta - b^2 \sin^2 \theta, \quad f = 2m\rho^2 - q^2 + 2abq\rho^2, \end{aligned} \quad (2.12)$$

and

$$\Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 + r^2) + q^2 + 2abq}{r^2} - 2m. \quad (2.13)$$

The physical quantities characterizing the solutions are given by [24, 25] :

$$Q = \frac{\pi}{4G} \frac{q}{(1 - a^2)(1 - b^2)}, \quad \mu = \frac{3qr_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \quad (2.14)$$

$$J_1 = \frac{\pi}{4G} \frac{2am + qb(1 + a^2)}{(1 - a^2)^2(1 - b^2)}, \quad \Omega_1 = \frac{a(r_+^2 + b^2)(r_+^2 + 1) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \quad (2.15)$$

$$J_2 = \frac{\pi}{4G} \frac{2bm + qa(1 + b^2)}{(1 - b^2)^2(1 - a^2)}, \quad \Omega_2 = \frac{b(r_+^2 + a^2)(r_+^2 + 1) + aq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq}, \quad (2.16)$$

$$S = \frac{\pi^2}{2G} \frac{(r_+^2 + a^2)(r_+^2 + b^2) + abq}{r_+(1 - a^2)(1 - b^2)}, \quad T = \frac{r_+^4(1 + a^2 + b^2 + 2r_+^2) - (ab + q)^2}{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}, \quad (2.17)$$

$$E = \frac{\pi}{4G} \frac{m(3 - a^2 - b^2 - a^2b^2) + 2qab(2 - a^2 - b^2)}{(1 - a^2)^2(1 - b^2)^2}. \quad (2.18)$$

The zeros of the function Δ_r give the locations of the horizons. Since $r^2\Delta_r$ is a cubic polynomial of r^2 , we can write

$$r^2\Delta_r = (r^2 - r_+^2)(r^2 - r_0^2)(r^2 - r_-^2). \quad (2.19)$$

We assume that $r_+^2 \geq r_0^2 \geq r_-^2$ by definition. Comparing (2.13) and (2.19), we note that

$$\begin{aligned} r_+^2 + r_0^2 + r_-^2 &= -(a^2 + b^2 + 1), \\ r_+^2 r_0^2 + r_0^2 r_-^2 + r_-^2 r_+^2 &= a^2 + b^2 + a^2 b^2 - 2m, \\ r_+^2 r_0^2 r_-^2 &= -(ab + q)^2. \end{aligned} \quad (2.20)$$

We can use (2.20) to show that the numerator of the expression for the temperature (2.17) can be rewritten as

$$r_+^2(r_+^2 - r_0^2)(r_+^2 - r_-^2). \quad (2.21)$$

Therefore, in the extremal case, $T = 0$, r_+^2 must coincide with r_0^2 , as one may have expected on general grounds.

In general, the solution (2.10 - 2.11) has four parameters, (m, q, a, b) . Extremality imposes a constraint, so m can be regarded as a function of a , b and q . We find it more convenient to use (r_+, a, b) instead of (q, a, b) as independent parameters. The reason is that while it is easy to solve (2.20) for q (with $r_0 = r_+$),

$$q = r_+^2 \sqrt{1 + a^2 + b^2 + 2r_+^2} - ab, \quad (2.22)$$

it is difficult to invert this relation to express r_+ in terms of q and a . The parameter m is also easily expressed as a function of a , b and r_+ ,

$$2m = 3r_+^4 + 2r_+^2(1 + a^2 + b^2) + a^2 + b^2 + a^2b^2. \quad (2.23)$$

Let us now discuss the allowed values of (r_+, a, b) . The temperature and entropy are non-negative if and only if r_+ is non-negative. Without loss of generality, we can assume that J_a , J_b and Q are all positive. It then follows that a , b and q should be positive. For the angular momenta J_a , J_b , charge Q and mass E to be finite, we also have $a^2, b^2 < 1$.

It was shown in [24] that the black hole is supersymmetric if and only if

$$r_+^2 = a + b + ab. \quad (2.24)$$

In the space of extremal black holes, there is a sense in which BPS black holes are the smallest ones. The Euclidean action for the black hole solution was computed in [25]:

$$I = \frac{\pi}{4G} \frac{T^{-1}}{(1 - a^2)(1 - b^2)} \left[m - (r_+^2 + a^2)(r_+^2 + b^2) - \frac{q^2 r_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \right]. \quad (2.25)$$

The black hole is thermodynamically stable compared to thermal AdS only if the Euclidean action is positive. This is analogous to the Hawking-Page phase transition [26] of AdS-Schwarzschild black holes. From (2.25), we find that the extremal limit of the critical surface $I = 0$ coincides with the BPS condition (2.24). For fixed values of a and b , the black hole is stable only if r_+^2 is greater than or equal to the BPS value (2.24).²

Figure 1 depicts the $a = b$ slice of the resulting domain. Only the region on the right side of the BPS curve is physical. The value of q in (2.22) is positive throughout the physical region. The mean free path l_{mfp} is found to be [14], for large r_+ ,

$$l_{\text{mfp}} \sim \left. \frac{S}{4\pi E} \right|_{\Omega=0} \propto \frac{1}{r_+}. \quad (2.26)$$

²Extra care should be taken for BPS black holes. It was shown in [27, 28, 29] that the correct thermodynamic variables are not the BPS values of Ω^a and μ , but their next to leading coefficients in a near extremal expansion. It was also shown that there is a phase transition in the (a, b) plane. We thank P. Silva for drawing our attention to these references.

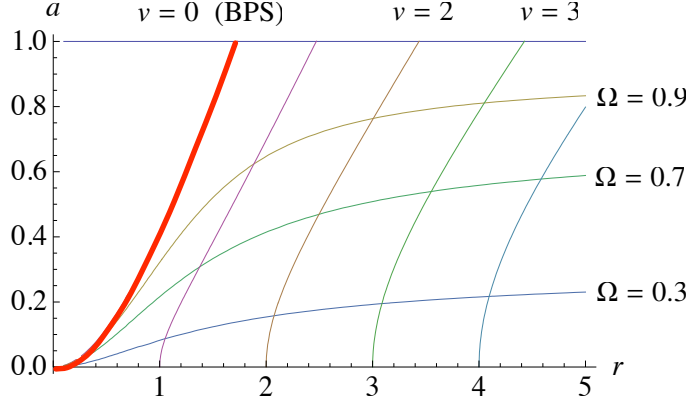


Figure 1: Moduli space of extremal black holes.

Thus we see that, as emphasized in [14], the BPS black holes lie on the opposite extreme of the “large black hole limit,” where the fluid mechanics becomes a good approximation.

One of the prediction from the fluid mechanics is that, in the large black hole limit, the entropy takes the form,

$$S = \frac{\pi}{4G} \frac{2\pi v^3}{(1 - \Omega_1^2)(1 - \Omega_2^2)}. \quad (2.27)$$

where v is inversely proportional to l_{mfp} up to a coefficient of order unity. In what follows, we will take v as a measure of the effective size of the black hole, a convenient notion especially when direct evaluation of l_{mfp} is not available. Figure 1 include the contours of fixed values of v as well as contours of fixed values of Ω . Note that the effective size of the BPS black holes vanishes, although their actual size can be arbitrarily large. The reason for the discrepancy is that all BPS black holes have $\Omega = 1$.

2.4 Near horizon geometry : a first look

For simplicity, we begin with the case where the two rotation parameters are equal, $a = b$. To make the $SU(2) \subset SO(4)$ symmetry manifest, we make the usual coordinate change,

$$\theta = \frac{\tilde{\theta}}{2}, \quad \phi_1 = \frac{\tilde{\psi} - \tilde{\phi}}{2}, \quad \phi_2 = \frac{\tilde{\psi} + \tilde{\phi}}{2}, \quad (2.28)$$

and introduce the invariant one-forms,

$$\sigma_1 = \cos \tilde{\psi} d\tilde{\theta} + \sin \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi}, \quad \sigma_2 = -\sin \tilde{\psi} d\tilde{\theta} + \cos \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi}, \quad \sigma_3 = d\tilde{\psi} + \cos \tilde{\theta} d\tilde{\phi}. \quad (2.29)$$

The near horizon limit of the extremal solution takes a remarkably simple form:

$$ds^2 = v_1^2 \left(-u^2 dt^2 + \frac{du^2}{u^2} \right) + v_2^2 [(\sigma_1)^2 + (\sigma_2)^2] + v_3^2 (\sigma_3 + \alpha u dt)^2, \quad (2.30)$$

$$A = e u dt + b \sigma_3, \quad (2.31)$$

where the near horizon parameters are given by

$$v_1^2 = \frac{r_+^2 + a^2}{4(3r_+^2 + 2a^2 + 1)}, \quad v_2^2 = \frac{r_+^2 + a^2}{4(1 - a^2)}, \quad v_3^2 = \left(\frac{(r_+^2 + a^2)^2 + a^2 q}{2r_+(r_+^2 + a^2)(1 - a^2)} \right)^2, \quad (2.32)$$

$$e = \frac{r_+ q}{2(r_+^2 + a^2)(3r_+^2 + 2a^2 + 1)}, \quad b = \frac{a q}{2(r_+^2 + a^2)(1 - a^2)}, \quad (2.33)$$

$$\alpha = \frac{a(1 - a^2)}{r_+(3r_+^2 + 2a^2 + 1)} \left[1 + \frac{2r_+^2 q}{(r_+^2 + a^2)^2 + a^2 q} \right]. \quad (2.34)$$

The large black hole limit in the sense discussed above is obtained by taking $r_+ \rightarrow \infty$ while keeping a fixed. In this limit, $q \rightarrow \sqrt{2}r_+^3$, and the near horizon data reduce to

$$v_1^2 = \frac{1}{12}, \quad v_2^2 = \frac{r_+^2}{4(1 - a^2)}, \quad v_3^2 = \frac{r_+^2}{4(1 - a^2)^2}, \quad (2.35)$$

$$e = \frac{\sqrt{2}}{6}, \quad b = \frac{\sqrt{2}ar_+}{2(1 - a^2)}, \quad \alpha = \frac{2\sqrt{2}a(1 - a^2)}{3r_+^2}. \quad (2.36)$$

The physical quantities also simplify:

$$Q = \frac{\pi}{4G} \frac{\sqrt{2}r_+^3}{(1 - a^2)^2}, \quad J = \frac{\pi}{4G} \frac{3ar_+^4}{(1 - a^2)^3}, \quad S = \frac{\pi}{4G} \frac{2\pi r_+^3}{(1 - a^2)^2}, \quad (2.37)$$

$$\mu = 3\sqrt{2}r_+, \quad \Omega = a, \quad E = \frac{\pi}{4G} \frac{3(a^2 + 3)r_+^4}{2(1 - a^2)^3}. \quad (2.38)$$

In the general case with $a \neq b$, the near horizon limit for arbitrary values of (r_+, a, b) is quite messy, so we focus on the large black hole limit. To the leading order in r_+ , the near horizon solution is given by ($c_\theta \equiv \cos \theta$, $s_\theta \equiv \sin \theta$)

$$ds^2 = \frac{1}{12} \left(-u^2 dt^2 + \frac{du^2}{u^2} \right) + \frac{r_+^2 d\theta^2}{1 - a^2 c_\theta^2 - b^2 s_\theta^2} + r_+^2 G_{ab}(\theta) D\phi^a D\phi^b, \quad (2.39)$$

$$A = \tilde{e} u dt + b_a(\theta) D\phi^a \quad (D\phi^a \equiv d\phi^a + \alpha^a u dt). \quad (2.40)$$

The $U(1)^2$ fibration part of the metric is rather non-trivial:

$$G_{ab} = (L^T \hat{G} L)_{ab}, \quad \hat{G} = \begin{pmatrix} 1 - a^2 c_\theta^2 & abs_\theta c_\theta \\ abs_\theta c_\theta & 1 - b^2 s_\theta^2 \end{pmatrix}, \quad L = \text{diag} \left(\frac{s_\theta}{1 - a^2}, \frac{c_\theta}{1 - b^2} \right). \quad (2.41)$$

The other parameters in the solution are given by

$$(\alpha^1, \alpha^2) = \frac{\sqrt{2}}{3r_+^2} (b(1 - a^2), a(1 - b^2)), \quad (2.42)$$

$$(b_1, b_2) = \sqrt{2}r_+ \left(\frac{as_\theta^2}{1 - a^2}, \frac{bc_\theta^2}{1 - b^2} \right), \quad \tilde{e} = \frac{\sqrt{2}}{6}. \quad (2.43)$$

The physical quantities in the large black hole limit are given by

$$Q = \frac{\pi}{4G} \frac{\sqrt{2}r_+^3}{(1 - a^2)(1 - b^2)}, \quad \mu = 3\sqrt{2}r_+, \quad (2.44)$$

$$J_1 = \frac{\pi}{4G} \frac{3ar_+^4}{(1 - a^2)^2(1 - b^2)}, \quad \Omega_1 = a, \quad (2.45)$$

$$S = \frac{\pi}{4G} \frac{2\pi r_+^3}{(1 - a^2)(1 - b^2)}, \quad E = \frac{\pi}{4G} \frac{3(3 - a^2 - b^2 - a^2 b^2)r_+^4}{2(1 - a^2)^2(1 - b^2)^2}. \quad (2.46)$$

3. Extremal black holes - I. equal rotation

We now begin our analysis of the near horizon geometry of black hole solutions in the class of supergravity theories reviewed in section 2. In this section, we will focus on the equal rotation case, in which the $SU(2)$ isometry simplifies the computation drastically, relegating the unequal rotation case to the next section.

3.1 Entropy function

The general near horizon solution of an extremal black hole in AdS_5 , which has an $SO(2,1) \times SU(2)$ invariance, take the following form: ³

$$ds^2 = v_1^2 \left(-u^2 dt^2 + \frac{du^2}{u^2} \right) + v_2^2 [(\sigma_1)^2 + (\sigma_2)^2] + v_3^2 (\sigma_3 + \alpha u dt)^2, \quad (3.1)$$

$$A^I = e^I u dt + b^I \sigma_3 = \tilde{e}^I u dt + b^I (\sigma_3 + \alpha u dt), \quad F^I = e^I du \wedge dt - b^I \sigma_1 \wedge \sigma_2, \quad (3.2)$$

$$X^I = u^I \quad (3.3)$$

We follow the standard procedure of the entropy function formalism [15, 16, 17]. First, we integrate the Lagrangian over the horizon to obtain the “near horizon action” :

$$\begin{aligned} \mathcal{F} \equiv & \frac{\pi}{G} v_1^2 v_2^2 v_3 \left(-\frac{2}{v_1^2} + \frac{2}{v_2^2} - \frac{v_3^2}{2v_2^4} + \frac{v_3^2 \alpha^2}{2v_1^4} - 2V + \frac{Q_{IJ} e^I e^J}{2v_1^4} - \frac{Q_{IJ} b^I b^J}{2v_2^4} \right) \\ & - \frac{\pi}{6G} C_{IJK} (3\tilde{e}^I + 2\alpha b^I) b^J b^K. \end{aligned} \quad (3.4)$$

The Chern-Simons term is slightly subtle. Inserting the ansatz (3.2) with constant \tilde{e}^I , b^I into the Lagrangian yields an incorrect result. To obtain the correct answer, one should consider b^I as a function of u , integrate by parts, and set b^I to be constant at the final stage.⁴ See appendix A for details.

The near horizon equations of motion can be derived by extremizing, \mathcal{F}

$$\frac{\partial \mathcal{F}}{\partial v_i} = 0, \quad \frac{\partial \mathcal{F}}{\partial b^I} = 0, \quad \frac{\partial \mathcal{F}}{\partial u^I} = 0, \quad (3.5)$$

while keeping (\tilde{e}^I, α) fixed. Explicitly, the equations read as follows.

1. Einstein equation:

$$\frac{1}{v_1^2} + \frac{v_3^2}{2v_2^4} - \frac{v_3^2 \alpha^2}{v_1^4} = \frac{Q_{IJ} e^I e^J}{2v_1^4}, \quad (3.6)$$

$$\frac{1}{v_2^2} - \frac{v_3^2}{v_2^4} + \frac{v_3^2 \alpha^2}{2v_1^4} = \frac{Q_{IJ} b^I b^J}{2v_2^4}, \quad (3.7)$$

$$-\frac{1}{v_1^2} + \frac{1}{v_2^2} = 2V. \quad (3.8)$$

³In the special case of the $U(1)^3$ theory, the entropy function for rotating AdS_5 black holes was previously obtained in [30, 31]. Here, we consider the large class of gauged supergravity theories all at once.

⁴We thank S. Trivedi and K. Goldstein for clarifying this point.

2. Maxwell equation:

$$Q_{IJ} [(v_1^{-2} v_2^2 v_3)(e^J \alpha - v_1^4 v_2^{-4} b^J)] - C_{IJK} e^J b^K = 0. \quad (3.9)$$

3. Scalar equation:

$$\frac{\partial Q_{IJ}}{\partial \varphi^i} (v_1^{-4} e^I e^J - v_2^{-4} b^I b^J) - 4 \frac{\partial V}{\partial \varphi^i} = 0. \quad (3.10)$$

Note that the scalar equation incorporates the constraint, $\frac{1}{6} C_{IJK} u^I u^J u^K = 1$, by means of a Lagrange multiplier. It is straightforward to confirm that the same equations follow from writing down the full equations of motion and inserting the near horizon ansatz.

The electric charges and the angular momentum are the conjugate variables of (\tilde{e}^I, α) with respect to \mathcal{F} ,

$$Q_I = \frac{\partial \mathcal{F}}{\partial \tilde{e}^I} = \frac{\pi}{G} \left[(v_1^{-2} v_2^2 v_3) Q_{IJ} e^J - \frac{1}{2} C_{IJK} b^J b^K \right], \quad (3.11)$$

$$J = \frac{\partial \mathcal{F}}{\partial \alpha} = \frac{\pi}{G} \left[(v_1^{-2} v_2^2 v_3) (v_3^2 \alpha + Q_{IJ} b^I e^J) - \frac{1}{3} C_{IJK} b^I b^J b^K \right]. \quad (3.12)$$

An alternative method to evaluate Q_I and J by using only the near horizon data in five dimensional supergravity was given in [32, 33] ($i_\xi \sigma_3 = -1$),

$$Q_I = \frac{1}{16\pi G} \int_{S_{\text{hor}}^3} (Q_{IJ} \star F^J + \frac{1}{2} C_{IJK} A^J \wedge F^K), \quad (3.13)$$

$$J = -\frac{1}{16\pi G} \int_{S_{\text{hor}}^3} [\star d\xi + (i_\xi A^I) (Q_{IJ} \star F^J + \frac{1}{3} C_{IJK} A^J \wedge F^K)]. \quad (3.14)$$

It is easy to check that the two methods give the same results.

The entropy function is the Legendre transform of the near horizon action:

$$\begin{aligned} \mathcal{E} &\equiv Q_I \tilde{e}^I + J \alpha - \mathcal{F} \\ &= \frac{\pi}{G} v_1^2 v_2^2 v_3 \left(\frac{2}{v_1^2} - \frac{2}{v_2^2} + \frac{v_3^2}{2v_2^4} + \frac{v_3^2 \alpha^2}{2v_1^4} + 2V + \frac{Q_{IJ} e^I e^J}{2v_1^4} + \frac{Q_{IJ} b^I b^J}{2v_2^4} \right). \end{aligned} \quad (3.15)$$

To complete the Legendre transformation, the variables (e^I, α) in \mathcal{E} should be eliminated in favor of the conjugates (Q_I, J) . Before doing so, we note that applying the Einstein equations (3.6-3.8) to (3.15) yields the area law as expected;

$$2\pi \mathcal{E} = \frac{4\pi^2}{G} v_2^2 v_3 = \frac{(\text{Area})_{\text{hor}}}{4G} = S_{\text{BH}}. \quad (3.16)$$

3.2 Large black hole solutions

Motivated by the large black hole limit of the minimal supergravity discussed in the previous section, we set

$$v_2^2 \equiv \frac{v^2}{4(1-a^2)}, \quad v_3^2 \equiv \frac{v^2}{4(1-a^2)^2}, \quad (3.17)$$

and consider the limit $v \gg 1$ with a fixed. We then try to solve the near horizon equations by expanding the other variables in powers of $1/v$.

3.2.1 Minimal supergravity revisited

As a warm up exercise, let us see how the minimal supergravity solution can be recovered from the near horizon equations. Setting $e^I = e\bar{X}^I$, $b^I = b\bar{X}^I$ in (3.6-3.9), the equations reduce to

$$\frac{1}{v_1^2} + \frac{v_3^2}{2v_2^4} - \frac{v_3^2\alpha^2}{v_1^4} = \frac{3e^2}{2v_1^4}, \quad (3.18)$$

$$\frac{1}{v_2^2} - \frac{v_3^2}{v_2^4} + \frac{v_3^2\alpha^2}{2v_1^4} = \frac{3b^2}{2v_2^4}, \quad (3.19)$$

$$\frac{1}{v_1^2} - \frac{1}{v_2^2} = 12. \quad (3.20)$$

$$(v_1^{-2}v_2^2v_3)(e\alpha - v_1^4v_2^{-4}b) = 2eb. \quad (3.21)$$

We can eliminate the α -dependent terms in (3.18) and (3.19),

$$\frac{1}{v_1^2} + \frac{2}{v_2^2} - \frac{3v_3^2}{2v_2^4} = \frac{3e^2}{2v_1^4} + \frac{3b^2}{v_2^4}. \quad (3.22)$$

Since the left hand side is $\mathcal{O}(1)$ in the leading order, we have two possibilities:

$$(a) \ e \sim \mathcal{O}(1), \ b \lesssim \mathcal{O}(v^2), \quad (b) \ e \lesssim \mathcal{O}(v^{-1}), \ b \sim \mathcal{O}(v^2). \quad (3.23)$$

For the possibility (b), we find from (3.18) and (3.19) that

$$\alpha = \pm \frac{v_1}{v_3}, \quad b = \pm \frac{v_2^2}{\sqrt{3}v_1}. \quad (3.24)$$

This “solution” cannot be the near horizon geometry of a rotating black hole. As $a \rightarrow 0$, the black hole horizon becomes a round three-sphere, and we expect the black hole to become static. But, in (3.24) the rotation parameter α and the magnetic dipole moment b retain non-zero values even when a vanishes.

The true solution, which comes from the possibility (a), can be summarized as follows:

$$v_1^2 = \frac{1}{12}, \quad v_2^2 = \frac{v^2}{4(1-a^2)}, \quad v_3^2 = \frac{v^2}{4(1-a^2)^2}, \quad (3.25)$$

$$e = \frac{\sqrt{2}}{6}, \quad b = \frac{\sqrt{2}av}{2(1-a^2)}, \quad \alpha = \frac{2\sqrt{2}a(1-a^2)}{3v^2}. \quad (3.26)$$

The physical quantities also can be easily computed:

$$Q = \frac{\pi}{4G} \frac{\sqrt{2}v^3}{(1-a^2)^2}, \quad J = \frac{\pi}{4G} \frac{3av^4}{(1-a^2)^3}, \quad S = \frac{\pi}{4G} \frac{2\pi v^3}{(1-a^2)^2}. \quad (3.27)$$

Of course, all the results agree perfectly with the large, extremal limit of the general solution we reviewed earlier.

3.2.2 General solution and universality

Since all supergravity theories contain the minimal supergravity, we expect the same v dependence of the near-horizon parameters in the large black hole limit. To the leading order in $1/v$, the equations take the following form.

$$2v_1^2 = -\frac{1}{V} = Q_{IJ}e^I e^J, \quad (3.28)$$

$$\frac{\partial Q_{IJ}}{\partial \varphi^i} e^I e^J = \frac{1}{V^2} \frac{\partial V}{\partial \varphi^i}, \quad (3.29)$$

$$\frac{v_2^2 v_3^2 \alpha^2}{2v_1^4} - \frac{Q_{IJ} b^I b^J}{2v_2^2} = \frac{v_3^2}{v_2^2} - 1, \quad (3.30)$$

$$\frac{v_2^2 v_3}{v_1^2} Q_{IJ} e^J \alpha = C_{IJK} e^J b^K. \quad (3.31)$$

Guided by the minimal supergravity solution, we introduce the reparametrization,

$$\alpha = \left(\frac{\bar{\alpha} v_1^2}{v^2} \right) 8\sqrt{2}a(1-a^2), \quad b^I = \bar{\alpha} \bar{b}^I \frac{\sqrt{2}av}{2(1-a^2)}. \quad (3.32)$$

In terms of the new variables, the equations (3.30, 3.31) read,

$$\bar{\alpha}^2(4 - Q_{IJ}\bar{b}^I \bar{b}^J) = 1, \quad (3.33)$$

$$Q_{IJ}e^J - \frac{1}{2}C_{IJK}e^J \bar{b}^K = 0, \quad (3.34)$$

which are completely independent of the parameter a . The other two equations (3.28, 3.29) are also independent of a . We conclude that the near horizon geometry of the large black holes in AdS_5 has a universal dependence on v and a , in agreement with the prediction of the fluid mechanics of the dual CFT [14].

Let us now discuss how to solve the remaining equations. In a theory with n vector fields, the extremal black hole solution (with equal rotation) depends on $n+1$ parameters. Physically, they correspond to n charges and the angular momentum. In solving the near-horizon equations, we will choose the parameters to be v , a and the $n-1$ independent values of φ^i .

In principle, it is easy to see how to solve the above equations. First, (3.28) and (3.29) give n quadratic equations for $\{e^I\}$, so that we can solve them to express $\{e^I\}$ as functions of $\{\varphi^i\}$. Second, (3.34) gives n linear equations for \bar{b}^I in terms of $\{e^I\}$ (and φ^i through Q_{IJ}). Finally, (3.33) determines $\bar{\alpha}$ in terms of \bar{b}^I .

In practice, the complete solution can be quite complicated because, in general, the equations for e^I lead to a polynomial equation of degree $2n$. Some theories may have extra symmetries to simplify the problem. Otherwise, one could look for a closed sub-family of solutions which effectively lowers the value of n . We will discuss such an example shortly.

3.2.3 $U(1)^3$ theory

To illustrate the algorithm to solve the near-horizon equations, we consider the $U(1)^3$ theory with $C_{123} = 1$. This theory has many simplifying features such as $u_I = (u^I)^{-1}$,

$Q_{IJ} = (u_I)^2 \delta_{IJ}$ and $V = -2(u_1 + u_2 + u_3)$. The scalar equations can be written as

$$2u_I + 2V^2 u_I^2 (e^I)^2 = -V \quad (3.35)$$

for each I (no sum). It is straightforward to solve all the equations, and the answer can be written as

$$v_1^2 = \frac{(L_1 L_2 L_3)^{-1/3}}{4 (\sum_I L_I^{-1})}, \quad \bar{\alpha} = \sqrt{\prod_I \frac{(1 - L_I)}{2L_I}}, \quad (3.36)$$

$$e^I = \frac{\sqrt{L_I^3 (1 - L_I)}}{2 (\sum_I L_I^{-1}) L_1 L_2 L_3}, \quad \bar{b}^I = \frac{\sqrt{L_I (1 - L_I)}}{\bar{\alpha} \sqrt{2} (L_1 L_2 L_3)^{1/3}}. \quad (3.37)$$

Here, the variables L_I satisfy $L_I > 0$ and $L_1 + L_2 + L_3 = 1$ (they are the same as X_i defined in section 6.2. of [14]). This parametrization makes it clear that the physical quantities

$$Q_I = \frac{\pi}{4G} \sqrt{\frac{1 - L_I}{L_I}} \frac{v^3}{(1 - a^2)^2}, \quad J = \frac{\pi}{4G} \frac{1}{(L_1 L_2 L_3)^{1/3}} \frac{2av^4}{(1 - a^2)^3}, \quad (3.38)$$

match perfectly with the $T \rightarrow 0$ limit of the results found in [34, 35, 14].⁵

3.2.4 A new solution

Next we consider an example of a theory of which no black hole solution is known in the literature. It has the following non-vanishing components of C_{IJK} ,

$$C_{123} = C_{234} = C_{341} = C_{412} = \frac{1}{4}. \quad (3.39)$$

This is the abelian truncation of the supergravity dual to the famous conifold CFT of [36]. The four $U(1)$ symmetries correspond to $U(1)_R$ symmetry, two mesonic symmetries and one baryonic symmetry. The scalars can be decomposed according to which vector multiplet they belong to:

$$X^1 = r + t + s_1, \quad X^2 = r - t + s_2, \quad X^3 = r + t - s_1, \quad X^4 = r - t - s_2. \quad (3.40)$$

The discrete symmetries of the theory makes it consistent to turn off the mesonic charges and relevant scalars identically, $s_1 = s_2 = 0$. The constraint $\frac{1}{6} C_{IJK} X^I X^J X^K = 1$ then becomes

$$r^3 - rt^2 = 1 \quad \implies \quad t = \sqrt{r^2 - \frac{1}{r}}, \quad (3.41)$$

where t is taken to be positive without loss of generality.

Following the general procedure described in the previous subsection, we first solve

$$Q_{IJ} e^I e^J = -\frac{1}{V}, \quad (\partial_r Q_{IJ}) e^I e^J = -\partial_r \left(\frac{1}{V} \right). \quad (3.42)$$

⁵To compare, for instance, our result with eq. (75) of [14], note that $\pi/4G = N^2/2$ in this theory and that v here is identified with $2\pi T(XYZ)^{1/3}/(X + Y + Z - 1)$ there.

In the (r, t) basis, $e^I = (e_r, e_t)$,

$$Q_{IJ} = \begin{pmatrix} Q_{rr} & Q_{rt} \\ Q_{tr} & Q_{tt} \end{pmatrix} = \begin{pmatrix} 4r^4 - 2r + r^{-2} & -4r^3 t \\ -4r^3 t & 4r^4 - 2r \end{pmatrix} \quad (3.43)$$

$$\partial_r Q_{IJ} = \begin{pmatrix} 16r^3 - 2 - 2r^{-3} & -2rt^{-1}(8r^3 - 5) \\ -2rt^{-1}(8r^3 - 5) & 16r^3 - 2 \end{pmatrix} \quad (3.44)$$

$$-\frac{1}{V} = \frac{4r^3 - 1}{18r^5}, \quad -\partial_r \left(\frac{1}{V} \right) = -\frac{8r^3 - 5}{18r^6}. \quad (3.45)$$

Equations (3.42) can be solved for (e_r, e_t) in a closed form, though the answer is rather complicated. The next step is to solve (3.34) :

$$Q_{IJe^J} - \frac{1}{2} C_{IJK} e^J \bar{b}^K = 0 \implies \begin{pmatrix} 3e_r & -e_t \\ -e_t & -e_r \end{pmatrix} \begin{pmatrix} \bar{b}_r \\ \bar{b}_t \end{pmatrix} = \begin{pmatrix} Q_{rr} & Q_{rt} \\ Q_{tr} & Q_{tt} \end{pmatrix} \begin{pmatrix} e_r \\ e_t \end{pmatrix}. \quad (3.46)$$

Again, the answer for (\bar{b}_r, \bar{b}_t) can be written down explicitly. Finally, (3.33) yields $\bar{\alpha}$ and complete the solution.

Although the explicit expressions for the near horizon solution are not very illuminating, this example clearly shows how the general method describe above can be used to solve all the near horizon equations.

4. Extremal black holes - II. Unequal rotation

4.1 Entropy function

Following [16, 37], we write down the near horizon ansatz for the most general rotating black holes in AdS_5 .⁶

$$ds^2 = w_1^2(\theta) \left(-u^2 dt^2 + \frac{du^2}{u^2} \right) + w_2^2(\theta) d\theta^2 + G_{ab}(\theta) (d\phi^a + \alpha^a u dt) (d\phi^b + \alpha^b u dt), \quad (4.1)$$

$$A^I = \tilde{e}^I u dt + b_a^I(\theta) (d\phi^a + \alpha^a u dt), \quad (4.2)$$

$$X^I = u^I(\theta). \quad (4.3)$$

In the metric, we have two functions w_1, w_2 and three components of G_{ab} . One combination out of these five can be removed by reparametrization of θ . The ansatz preserves $SO(2, 1)$ symmetry realized by the Killing vectors,⁷

$$L_{+1} = \partial_t, \quad L_0 = t\partial_t - u\partial_u, \quad L_{-1} = \frac{1}{2}(1/u^2 + t^2)\partial_t - (tu)\partial_u - (\alpha^a/u)\partial_{\phi^a}. \quad (4.4)$$

⁶The entropy function of general rotating black holes in $5d$ ungauged supergravity was first considered in [37]. To our knowledge, the results in gauged supergravity presented here are new, apart from the general near horizon analysis for *supersymmetric* black holes performed from a different angle in [38, 39].

⁷It was proven in [40] that the near horizon geometry of any extremal black hole in four and five dimensions, in a generic second order theory of gravity coupled to uncharged scalars and gauge fields (including a cosmological constant) must have $SO(2, 1)$ symmetry.

In computing the near horizon action, the Einstein-Hilbert term deserves some comments:

$$R = R_1 + R_2 + R_3, \quad (4.5)$$

$$R_1 = -\frac{2}{w_1^2} - \frac{2\dot{w}_1^2}{w_1^2 w_2^2} + \frac{4\dot{w}_1 \dot{w}_2}{w_1 w_2^3} - \frac{4\ddot{w}_1}{w_1 w_2^2}, \quad R_2 = \frac{G_{ab} \alpha^a \alpha^b}{2w_1^4}, \quad (4.6)$$

$$R_3 = \frac{(G^{ab} \partial_\theta G_{ba})^2 - (G^{ab} \partial_\theta G_{bc})(G^{cd} \partial_\theta G_{da})}{4w_2^2} - \frac{2\partial_\theta(w_1^2 w_2^{-1} \partial_\theta \sqrt{G})}{\sqrt{G} w_1^2 w_2}. \quad (4.7)$$

The metric ansatz (4.1) takes the form of a two-torus (ϕ^1, ϕ^2) fibered over the “base” space (t, u, θ) . The first term is simply the scalar curvature of the base space, where dots denote derivatives in θ . The last term is the contribution from the fiber metric. The second term can be thought of as a “Kaluza-Klein” electric flux density. Combining with

$$\sqrt{-g} = w_1^2 w_2 \sqrt{G}, \quad (4.8)$$

we find

$$\begin{aligned} \sqrt{-g}R = & w_1^2 w_2 \sqrt{G} \left(-\frac{2}{w_1^2} + \frac{2\dot{w}_1^2}{w_1^2 w_2^2} + \frac{4\dot{w}_1 \partial_\theta \sqrt{G}}{w_1 w_2^2 \sqrt{G}} + \frac{1}{2} G_{ab} \alpha^a \alpha^b \right) \\ & + w_1^2 w_2 \sqrt{G} \left(\frac{1}{w_2^2} \left(\frac{\partial_\theta \sqrt{G}}{\sqrt{G}} \right)^2 + \frac{\partial_\theta G^{ab} \partial_\theta G_{ab}}{4w_2^2} \right) - 2\partial_\theta \left(\frac{1}{w_2} \partial_\theta (w_1^2 \sqrt{G}) \right). \end{aligned} \quad (4.9)$$

The last term, being a total derivative, does not affect the equations of motion. However, we will show that it makes a non-zero contribution to the entropy function.

Adding the matter contributions and evaluating the Chern-Simons term, we obtain the near horizon action,⁸

$$\begin{aligned} \mathcal{F} = & \frac{\pi}{4G_5} \int d\theta \sqrt{G} w_1^2 w_2 \left(-\frac{2}{w_1^2} - 2V + \frac{G_{ab} \alpha^a \alpha^b}{2w_1^4} \right) - \frac{\pi}{4G_5} \left[\frac{2}{w_2} \partial_\theta (w_1^2 \sqrt{G}) \right]_0^{\pi/2} \\ & + \frac{\pi}{4G_5} \int d\theta \sqrt{G} w_1^2 w_2 \left(\frac{2\dot{w}_1^2}{w_1^2 w_2^2} + \frac{4\dot{w}_1 \partial_\theta \sqrt{G}}{w_1 w_2^2 \sqrt{G}} + \frac{1}{w_2^2} \left(\frac{\partial_\theta \sqrt{G}}{\sqrt{G}} \right)^2 + \frac{\partial_\theta G^{ab} \partial_\theta G_{ab}}{4w_2^2} \right) \\ & + \frac{\pi}{4G_5} \int d\theta \sqrt{G} w_1^2 w_2 \left(\frac{Q_{IJ} e^I e^J}{2w_1^4} - \frac{Q_{IJ} G^{ab} \partial_\theta b_a^I \partial_\theta b_b^J}{2w_2^2} - \frac{g_{ij} \partial_\theta \varphi^i \partial_\theta \varphi^j}{2w_2^2} \right) \\ & - \frac{\pi}{4G_5} \int d\theta \left(\frac{1}{6} C_{IJK} (3\tilde{e}^I + 2\alpha^a b_a^I) \epsilon^{bc} b_b^J \partial_\theta b_c^K \right), \end{aligned} \quad (4.10)$$

where $\epsilon^{12} = -1 = -\epsilon^{21}$. Upon taking a variation, we obtain the near horizon equations:

1. Einstein equation

$$(a) \quad \frac{w_1}{4} \frac{\delta \mathcal{F}}{\delta w_1} + \frac{w_2}{2} \frac{\delta \mathcal{F}}{\delta w_2} = 0 :$$

⁸In this subsection, we denote the Newton’s constant by G_5 to avoid confusion with $G \equiv \det(G_{ab})$.

$$\frac{1}{w_1^2} + 2V + \frac{1}{\sqrt{G}w_1^2w_2} \partial_\theta \left(\frac{w_1}{w_2} \partial_\theta (\sqrt{G}w_1) \right) = 0. \quad (4.11)$$

$$(b) \quad \frac{w_1}{4} \frac{\delta \mathcal{F}}{\delta w_1} + \frac{w_2}{2} \frac{\delta \mathcal{F}}{\delta w_2} + G^{ab} \frac{\delta \mathcal{F}}{\delta G^{ab}} = 0 : \\ - \frac{1}{w_1^2} - \frac{\dot{w}_1^2}{w_1^2 w_2^2} - \frac{3}{w_1 w_2} \partial_\theta \left(\frac{\dot{w}_1}{w_2} \right) + \frac{\dot{w}_1 \partial_\theta \sqrt{G}}{w_2^2 w_1 \sqrt{G}} + \frac{1}{w_2^2} \left(\frac{\partial_\theta \sqrt{G}}{\sqrt{G}} \right)^2 \\ + \frac{\partial_\theta G^{ab} \partial_\theta G_{ab}}{4w_2^2} + \frac{G_{ab} \alpha^a \alpha^b}{w_1^4} + \frac{Q_{IJ} e^I e^J}{2w_1^4} - \frac{g_{ij} \partial_\theta \varphi^i \partial_\theta \varphi^j}{2w_2^2} = 0. \quad (4.12)$$

$$(c) \quad w_2 \frac{\delta \mathcal{F}}{\delta w_2} + G^{ab} \frac{\delta \mathcal{F}}{\delta G^{ab}} = 0 : \\ - \frac{4}{w_1 w_2} \partial_\theta \left(\frac{\dot{w}_1}{w_2} \right) + \frac{2\dot{w}_1 \partial_\theta \sqrt{G}}{w_2^2 w_1 \sqrt{G}} - \frac{1}{w_2} \partial_\theta \left(\frac{\partial_\theta \sqrt{G}}{w_2 \sqrt{G}} \right) + \frac{1}{w_2^2} \left(\frac{\partial_\theta \sqrt{G}}{\sqrt{G}} \right)^2 \\ + \frac{\partial_\theta G^{ab} \partial_\theta G_{ab}}{2w_2^2} + \frac{G_{ab} \alpha^a \alpha^b}{2w_1^4} - \frac{Q_{IJ} G^{ab} \partial_\theta b_a^I \partial_\theta b_b^J}{2w_2^2} - \frac{g_{ij} \partial_\theta \varphi^i \partial_\theta \varphi^j}{w_2^2} = 0. \quad (4.13)$$

$$(d) \quad \frac{\delta \mathcal{F}}{\delta G^{ab}} - \frac{1}{2} \left(G^{cd} \frac{\delta \mathcal{F}}{\delta G^{cd}} \right) G_{ab} = 0 : \\ X_{ab} - \frac{1}{2} (G^{cd} X_{cd}) G_{ab} = 0, \\ X_{ab} \equiv \frac{1}{\sqrt{G}w_1^2w_2} \left[\partial_\theta \left(\frac{\sqrt{G}w_1^2}{w_2} \partial_\theta G_{ab} \right) - G_{ac} \partial_\theta \left(\frac{\sqrt{G}w_1^2}{w_2} \partial_\theta G^{cd} \right) G_{db} \right] \\ + \frac{2(G_{ac} \alpha^c)(G_{bd} \alpha^d)}{w_1^4} + \frac{2Q_{IJ} \partial_\theta b_a^I \partial_\theta b_b^J}{w_2^2}. \quad (4.14)$$

2. Maxwell equation

$$\frac{\sqrt{G}w_2}{w_1^2} Q_{IJ} e^J \alpha^a + \partial_\theta \left(\frac{\sqrt{G}w_1^2}{w_2} Q_{IJ} G^{ab} \partial_\theta b_b^J \right) - C_{IJK} e^J \epsilon^{ab} \partial_\theta b_b^K = 0. \quad (4.15)$$

3. Scalar equation

$$\frac{2}{\sqrt{G}w_1^2w_2} \partial_\theta \left(\frac{\sqrt{G}w_1^2}{w_2} g_{ij} \partial_\theta \varphi^j \right) + \frac{\partial Q_{IJ}}{\partial \varphi^i} \left(\frac{e^I e^J}{w_1^4} - \frac{G^{ab} \partial_\theta b_a^I \partial_\theta b_b^J}{w_2^2} \right) - 4 \frac{\partial V}{\partial \varphi^i} = 0 \quad (4.16)$$

The conserved charges are given by

$$Q_I = \frac{\delta \mathcal{F}}{\delta \tilde{e}^I} = \frac{\pi}{4G_5} \int d\theta \left(\frac{\sqrt{G}w_2}{w_1^2} Q_{IJ} e^J - \frac{1}{2} C_{IJK} \epsilon^{ab} b_a^J \partial_\theta b_b^K \right), \quad (4.17)$$

$$J_a = \frac{\delta \mathcal{F}}{\delta \alpha^a} = \frac{\pi}{4G_5} \int d\theta \left[\frac{\sqrt{G}w_2}{w_1^2} (G_{ab} \alpha^b + Q_{IJ} e^I b_a^J) - \frac{1}{3} C_{IJK} b_a^I \epsilon^{bc} b_b^J \partial_\theta b_c^K \right]. \quad (4.18)$$

Finally, the entropy function is computed to be

$$\begin{aligned}
\mathcal{E} &= Q_I \tilde{e}^I + J_a \alpha^a - \mathcal{F} \\
&= \frac{\pi}{4G_5} \int d\theta \sqrt{G} w_1^2 w_2 \left(\frac{2}{w_1^2} + 2V + \frac{G_{ab} \alpha^a \alpha^b}{2w_1^4} \right) + \frac{\pi}{4G_5} \left[\frac{2}{w_2} \partial_\theta \left(w_1^2 \sqrt{G} \right) \right]_0^{\pi/2} \\
&\quad - \frac{\pi}{4G_5} \int d\theta \sqrt{G} w_1^2 w_2 \left(\frac{2\dot{w}_1^2}{w_1^2 w_2^2} + \frac{4\dot{w}_1 \partial_\theta \sqrt{G}}{w_1 w_2^2 \sqrt{G}} + \frac{1}{w_2^2} \left(\frac{\partial_\theta \sqrt{G}}{\sqrt{G}} \right)^2 + \frac{\partial_\theta G^{ab} \partial_\theta G_{ab}}{4w_2^2} \right) \\
&\quad + \frac{\pi}{4G_5} \int d\theta \sqrt{G} w_1^2 w_2 \left(\frac{Q_{IJ} e^I e^J}{2w_1^4} + \frac{Q_{IJ} G^{ab} \partial_\theta b_a^I \partial_\theta b_b^J}{2w_2^2} + \frac{g_{ij} \partial_\theta \varphi^i \partial_\theta \varphi^j}{2w_2^2} \right). \tag{4.19}
\end{aligned}$$

Using the equations of motion, we can easily confirm the area law for the black hole entropy:

$$2\pi\mathcal{E} = \frac{\pi^2}{G_5} \int d\theta \left(w_2 \sqrt{G} \right) + \frac{\pi^2}{2G_5} \left[\frac{\sqrt{G} \partial_\theta (w_1^2)}{w_2} \right]_0^{\pi/2} = \frac{(\text{Area})_{\text{hor}}}{4G_5} = S_{\text{BH}}. \tag{4.20}$$

The boundary term does not contribute to the entropy because both \sqrt{G} and $\partial_\theta(w_1^2)$ must vanish at $\theta = 0, \pi/2$ to avoid singular geometry. To reach this conclusion, it is essential to keep both contributions to the boundary term, one coming from (4.9) and the other from (4.11).

4.2 Large black hole solutions

To find a large black hole solution to the near horizon equations, we need to determine the dependence of the near horizon parameters on the size parameter v . For the non-derivative terms, we expect that the results from the equal rotation case continue to hold.:

$$w_1 \sim 1, \quad w_2 \sim v, \quad w_3 \sim v, \quad e^I \sim 1, \quad \alpha^a \sim v^{-2}, \quad b_a^I \sim v. \tag{4.21}$$

The derivatives $\partial_\theta w_2$, $\partial_\theta w_3$ and $\partial_\theta b_a^I$, are already present in the minimal supergravity solution, and we again assume the same dependence since all supergravity theories contain the minimal one:

$$\partial_\theta w_2/w_2 \sim 1, \quad \partial_\theta w_3/w_3 \sim 1, \quad \partial_\theta b_a^I \sim v. \tag{4.22}$$

The remaining two terms, $\partial_\theta w_1$ and $\partial_\theta \varphi^i$, are somewhat subtle. In minimal supergravity both of them are strictly zero, but in general they are expected to be non-zero. The two should be of the same order in v , as can be seen from, for example,

$$2w_1^2 = -\frac{1}{V} = Q_{IJ} e^I e^J, \tag{4.23}$$

which is obtained by taking the leading terms of the Einstein equations (4.11, 4.12). Very few explicit solutions with multiple charges and arbitrary rotations are known in the literature; refs. [41] and [42] are the only examples we are aware of. In these references, all the θ -dependences of w_1 and φ^i come in through the combination $\rho^2 = r^2 + a^2 c_\theta^2 + b^2 s_\theta^2$, which implies that,

$$\partial_\theta w_1 \sim \partial_\theta \varphi^i \sim \frac{2s_\theta c_\theta (b^2 - a^2)}{v^2}. \tag{4.24}$$

We will proceed by assuming that this $1/v^2$ suppression is true of all large black hole solutions and later confirm that this assumption is self-consistent.

It is now straightforward to examine the near horizon equations to the order equations in $1/v$. The first two components of the Einstein equations as well as the scalar and Maxwell equations become quite simple.

$$2w_1^2 = -\frac{1}{V} = Q_{IJ}e^I e^J, \quad (4.25)$$

$$\frac{\partial Q_{IJ}}{\partial \varphi^i} \left(\frac{e^I e^J}{w_1^4} \right) - 4 \left(\frac{\partial V}{\partial \varphi^i} \right) = 0, \quad (4.26)$$

$$\frac{\sqrt{G}w_2}{w_1^2} Q_{IJ}e^J \alpha^a - C_{IJK}e^J \epsilon^{ab} \partial_\theta b_b^K = 0, \quad (4.27)$$

The remaining equations (4.13) and (4.14) are also slightly simplified,

$$-\frac{\partial_\theta}{w_2} \left(\frac{\partial_\theta \sqrt{G}}{w_2 \sqrt{G}} \right) + \frac{1}{w_2^2} \left(\frac{\partial_\theta \sqrt{G}}{\sqrt{G}} \right)^2 + \frac{\partial_\theta G^{ab} \partial_\theta G_{ab}}{2w_2^2} + \frac{G_{ab} \alpha^a \alpha^b}{2w_1^4} - \frac{Q_{IJ} G^{ab} \partial_\theta b_a^I \partial_\theta b_b^J}{2w_2^2} = 0. \quad (4.28)$$

$$\begin{aligned} X_{ab} - \frac{1}{2}(G^{cd} X_{cd}) G_{ab} &= 0, \\ X_{ab} &\equiv \frac{1}{\sqrt{G}w_2} \left[\partial_\theta \left(\frac{\sqrt{G}}{w_2} \partial_\theta G_{ab} \right) - G_{ac} \partial_\theta \left(\frac{\sqrt{G}}{w_2} \partial_\theta G^{cd} \right) G_{db} \right] \\ &\quad + \frac{2(G_{ac} \alpha^c)(G_{bd} \alpha^d)}{w_1^4} + \frac{2Q_{IJ} \partial_\theta b_a^I \partial_\theta b_b^J}{w_2^2}. \end{aligned} \quad (4.29)$$

The prediction from the fluid mechanics and the universal solution in the equal rotation case obtained earlier lead us to look for a universal solution even in the unequal rotation case. In fact, a straightforward computation shows that the following general solution satisfies all the leading order near horizon equations:

$$ds^2 = v_1^2 \left(-u^2 dt^2 + \frac{du^2}{u^2} \right) + \frac{v^2 d\theta^2}{1 - a^2 c_\theta^2 - b^2 s_\theta^2} + v^2 G_{ab}(\theta) D\phi^a D\phi^b, \quad (4.30)$$

$$A^I = \tilde{e}^I u dt + b_a^I(\theta) D\phi^a \quad (D\phi^a \equiv d\phi^a + \alpha^a u dt), \quad (4.31)$$

where

$$(\alpha^1, \alpha^2) = \frac{4\sqrt{2}\bar{\alpha}v_1^2}{v^2} (b(1-a^2), a(1-b^2)), \quad (4.32)$$

$$(b_1^I, b_2^I) = \sqrt{2}\bar{\alpha}\bar{b}^I v \left(\frac{as_\theta^2}{1-a^2}, \frac{bc_\theta^2}{1-b^2} \right). \quad (4.33)$$

All the constants $(v_1, e^I, \bar{\alpha}, \bar{b}^I, \varphi^i)$ are the same as in the previous section (3.33, 3.34). The form of the metric, apart from the values of α^a , has been taken from the minimal supergravity solution (2.41).

5. Discussions

5.1 Local entropy and charge densities

The universal rotation dependence predicted by fluid mechanics [14] determines not only the total entropy and charges but also their local density. Let us check how our computations on the horizon compare with the fluid mechanics on the boundary.

From the solution (4.30), we find the entropy and angular momentum densities

$$\frac{dS}{\sin \theta \cos \theta d\theta} = \frac{S_0}{(1-a^2)(1-b^2)}, \quad \frac{dJ_1}{\sin \theta \cos \theta d\theta} = \frac{(a \sin^2 \theta) J_0}{(1-a^2)^2(1-b^2)}. \quad (5.1)$$

with S_0 and J_0 independent of θ . The charges Q_I have the same angle dependence as S . The coordinates of [24] we have been using is related to the manifestly asymptotically AdS coordinate by [14]⁹

$$\tilde{r}^2 \sin^2 \chi = \frac{(r^2 + a^2) \sin^2 \theta}{1 - a^2}, \quad \tilde{r}^2 \cos^2 \chi = \frac{(r^2 + b^2) \cos^2 \theta}{1 - b^2}. \quad (5.2)$$

When $r, \tilde{r} \gg 1$, we can eliminate them to obtain the relation between the angles,

$$\cos^2 \theta = \frac{(1 - b^2) \cos^2 \chi}{(1 - a^2 \sin^2 \chi - b^2 \cos^2 \chi)}. \quad (5.3)$$

In terms of the χ coordinate, the local densities take the form,

$$\begin{aligned} \frac{dS}{\sin \chi \cos \chi d\chi} &= \frac{S_0}{(1 - a^2 \sin^2 \chi - b^2 \cos^2 \chi)^2}, \\ \frac{dJ_1}{\sin \chi \cos \chi d\chi} &= \frac{(a \sin^2 \chi) J_0}{(1 - a^2 \sin^2 \chi - b^2 \cos^2 \chi)^3}, \end{aligned} \quad (5.4)$$

which coincides precisely with the expressions obtained from the fluid mechanics in [14].

Note that this agreement is not quite trivial. The fluid mechanics computation was performed over the round S^3 at the boundary of the AdS, while our computation was done on the black hole horizon which has the shape of a squashed S^3 due to the rotations. In general, the map between a region of the fluid and the corresponding region on the horizon could be quite complicated. Our result above shows that such a complication does not occur to the leading order in the large black hole limit.

5.2 The h function

The entropy function formalism enables us to express the entropy S , angular momenta J_a and electric charges Q_I in terms of the near horizon data without ever referring to the full solution. In contrast, the mass E , angular velocity ω_a and the electric potential μ^I cannot be computed in general from the near horizon data only.

⁹The coordinate change is taken from eq. (141) of [14], which uses different names for the variables $(r, \theta; \tilde{r}, \chi)_{\text{here}} = (y, \tilde{\theta}; r, \theta)_{\text{there}}$ and has a typographical error which we correct here.

In the large black hole limit, however, the simple mapping to the fluid mechanics we have discussed so far enables us to write down (E, Ω^a, μ^I) in terms of the near horizon data. First, by comparing

$$S = \left(\frac{\pi}{4G}\right) \frac{2\pi v^3}{(1-a^2)(1-b^2)}, \quad (5.5)$$

$$Q_I = \left(\frac{\pi}{4G}\right) \frac{v^3 Q_{IJ} e^J}{2v_1^2(1-a^2)(1-b^2)}, \quad (5.6)$$

$$J_1 = \left(\frac{\pi}{4G}\right) \frac{\sqrt{2}v^4 Q_{IJ} e^I \bar{b}^J \bar{\alpha}}{8v_1^2(1-a^2)(1-b^2)} \left[\frac{2a}{1-a^2} \right], \quad (5.7)$$

with the predictions of fluid mechanics, we note the identification,

$$\Omega_1 = a, \quad \Omega_2 = b, \quad (5.8)$$

and the $T \rightarrow 0$ limit of the h function,

$$T^4 h \rightarrow \frac{\sqrt{2}v^4 Q_{IJ} e^I \bar{b}^J \bar{\alpha}}{64\pi G v_1^2}, \quad T^3 \partial_I h \rightarrow \frac{v^3 Q_{IJ} e^J}{16\pi G v_1^2}, \quad T^3 (4h - \nu^I \partial_I h) \rightarrow \frac{2\pi v^3}{8\pi G}. \quad (5.9)$$

Next, since both J_a and E are proportional to the function h with a universal rotation dependence, we have

$$E = \left(\frac{\pi}{4G}\right) \frac{\sqrt{2}v^4 Q_{IJ} e^I \bar{b}^J \bar{\alpha}}{8v_1^2(1-a^2)(1-b^2)} \left[\frac{2a^2}{1-a^2} + \frac{2b^2}{1-b^2} + 3 \right]. \quad (5.10)$$

Note the estimate for the mean free path

$$l_{\text{mfp}} \sim \frac{S}{4\pi E} \Big|_{\Omega=0} \sim \frac{1}{v} \left(\frac{v_1^2}{Q_{IJ} e^I \bar{b}^J \bar{\alpha}} \right), \quad (5.11)$$

where the quantity in the parenthesis is generically of order one. So we confirm that the fluid dynamics is indeed a good description in the limit of large v .

Finally, the chemical potentials are obtained by noting that, in the absence of rotation, $E = \frac{3}{4}\mu^I Q_I$ should hold (See section 2.5 of [14]):

$$\mu^I = \sqrt{2}v \bar{\alpha} \bar{b}^I. \quad (5.12)$$

All gauged supergravity theories contain the minimal supergravity as a closed subsector. The minimal supergravity admit large extremal black hole solutions. It is reasonable to expect that, generically, small deformation away from the minimal supergravity will not change the properties of the black hole drastically, as we confirmed in the two examples above. Thus, large extremal black holes are rather generic objects in gauged supergravity.

From the CFT point of view, this means that the zero temperature limit of the CFT “fluid” is smooth; all physical quantities $(S, J_a, Q_I; E, \Omega_a, \mu^I)$ remain finite as T approaches zero. In order for E and J_a to have a smooth limit as $\nu_I = \mu_I/T$ become large, the $h(\nu)$ function should approach a homogeneous function of degree four in the leading order:

$$h(\nu) = h_4(\nu) + (\text{sub-leading}), \quad h_4(\lambda\nu) = \lambda^4 h(\nu). \quad (5.13)$$

Recall that the entropy S is proportional to $T^3(4h - \nu_I \partial_I h)$. The leading term $h_4(\nu)$ does not contribute to S . The only possible contribution to S could come from a homogeneous function of degree three,

$$h(\nu) = h_4(\nu) + h_3(\nu) + (\text{sub-leading}), \quad h_k(\lambda\nu) = \lambda^k h(\nu). \quad (5.14)$$

In the $U(1)^3$ theory, up to an overall normalization, the h function and its “descendants” are given by [14]

$$h = \frac{L_1 L_2 L_3}{(L_1 + L_2 + L_3 - 1)^4}, \quad \nu_I = \frac{\sqrt{L_I(1 - L_I)}}{L_1 + L_2 + L_3 - 1}, \quad (5.15)$$

$$\partial_I h = \frac{2L_1 L_2 L_3}{(L_1 + L_2 + L_3 - 1)^3} \sqrt{\frac{1 - L_I}{L_I}}, \quad 4h - \nu^I \partial_I h = \frac{2L_1 L_2 L_3}{(L_1 + L_2 + L_3 - 1)^3}. \quad (5.16)$$

In the $T \rightarrow 0$ limit, the three variables L_I approaches the “extremal triangle” ($\sum_I L_I = 1, L_I > 0$). In this limit, the leading behavior of the h -function is manifestly quartic in ν and it is straightforward to separate $h_4(\nu)$ explicitly:

$$h_4(\nu) = \frac{1}{4} [2(\nu_1^2 \nu_2^2 + \nu_2^2 \nu_3^2 + \nu_3^2 \nu_1^2) - (\nu_1^4 + \nu_2^4 + \nu_3^4)] \quad (5.17)$$

$$= \frac{1}{4} (\nu_1 + \nu_2 + \nu_3)(-\nu_1 + \nu_2 + \nu_3)(\nu_1 - \nu_2 + \nu_3)(\nu_1 + \nu_2 - \nu_3). \quad (5.18)$$

It is also not difficult to extract $h_3(\nu)$:

$$h_3(\nu) = \sqrt{(-\nu_1^2 + \nu_2^2 + \nu_3^2)(\nu_1^2 - \nu_2^2 + \nu_3^2)(\nu_1^2 + \nu_2^2 - \nu_3^2)}/2. \quad (5.19)$$

It would be interesting to understand further the functions h_4 and h_3 from both the supergravity side and the dual CFT side. From the supergravity point of view, these functions can depend only on the parameters of the Lagrangian, namely, C_{IJK} and \bar{X}_I . It follows that, if h_4 is a polynomial, then the coefficients in the expansion,

$$h_4(\nu) = \frac{1}{4!} h_{IJKL} \nu^I \nu^J \nu^K \nu^L, \quad (5.20)$$

should be constants composed of C_{IJK} and \bar{X}_I . Unfortunately, it appears that h_4 is not a polynomial in general. We have checked whether the h_4 of the conifold CFT example discussed in section 3.2.4 can be expressed as a polynomial in ν^I and found the answer in the negative.

5.3 Future directions

The near horizon analysis performed in this paper should be regarded as a first step to reveal the connection between the two holographic screens, namely, the black hole horizon and the AdS boundary. There are several directions one may pursue further.

We restricted our attention to the extremal black holes only. It will be clearly useful to do similar computations for black holes with non-zero temperature and make a more comprehensive comparison with the fluid dynamics. Non-extremal entropy function of [43] may be useful in this regard.

The entropy function we obtained could be used away from the large black hole limit. For instance, it could be used to explore the existence of supersymmetric black holes in theories where no solutions have been constructed.

Finally, it would be interesting to compute the subleading corrections both in the fluid mechanics and on the near horizon equations and see how the map between the two changes. Recently, some progress in this direction was reported in [44], where it was shown, to the first subleading order, how the region of the fluid evolves in the radial direction in the case of uncharged black branes. Incorporating electric charges and working in the global AdS rather than the Poincaré patch, the corrections in the fluid mechanics could be matched against corresponding corrections to the near horizon equations we obtained in this paper. The systematic derivative expansion developed in [44, 45, 46, 47, 48] would be helpful in such an attempt. We hope to return to some of these questions in the near future.

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Appendix

A. Chern-Simons contribution to the entropy function

A very general and thorough discussion of how to deal with Chern-Simons contributions to the entropy function can be found in [17]. In our case, in essence, the correct procedure amounts to maintaining the radial dependence of the variables b^I in the near horizon ansatz,

$$\begin{aligned} A^I &= \tilde{e}^I u dt + b^I (\sigma_3 + \alpha u dt) \equiv \tilde{e}^I u dt + b^I \tilde{\sigma}_3, \\ F^I &= (\tilde{e}^I + \alpha b^I) du \wedge dt - b^I \sigma_1 \wedge \sigma_2 + (\partial_u b^I) du \wedge \tilde{\sigma}_3, \end{aligned} \tag{A.1}$$

in the intermediate steps and setting b^I to constant only at the final step.

Inserting (A.1) into the Chern-Simons term of the Lagrangian,

$$(16\pi G)\mathcal{L}_{\text{CS}} = -\frac{1}{6}C_{IJK}F^I \wedge F^J \wedge A^K, \tag{A.2}$$

we find

$$(16\pi G) \mathcal{L}_{\text{CS}} = (16\pi G) (\mathcal{L}_{\text{CS}}^1 + \mathcal{L}_{\text{CS}}^2) , \quad (\text{A.3})$$

$$\begin{aligned} (16\pi G) \mathcal{L}_{\text{CS}}^1 &= \frac{1}{3} C_{IJK} (b^I \tilde{\sigma}_3) ((\tilde{e}^J + \alpha b^J) du dt) (b^K \sigma_1 \sigma_2) \\ &= -\frac{1}{3} C_{IJK} (\tilde{e}^I b^J b^K + \alpha b^I b^J b^K) (dt du \sigma_1 \sigma_2 \tilde{\sigma}_3) , \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} (16\pi G) \mathcal{L}_{\text{CS}}^2 &= \frac{1}{3} C_{IJK} (\tilde{e}^I u dt) (\partial_u b^J du \tilde{\sigma}_3) (b^K \sigma_1 \sigma_2) \\ &= \frac{1}{6} C_{IJK} \tilde{e}^I u \partial_u (b^J b^K) (dt du \sigma_1 \sigma_2 \tilde{\sigma}_3) \\ &= -\frac{1}{6} C_{IJK} \tilde{e}^I b^J b^K (dt du \sigma_1 \sigma_2 \tilde{\sigma}_3) + (\text{total derivative}). \end{aligned} \quad (\text{A.5})$$

Adding the two terms and discarding the total derivative term, we obtain

$$(16\pi G) \mathcal{L}_{\text{CS}} = -C_{IJK} \left(\frac{1}{2} \tilde{e}^I b^J b^K + \frac{1}{3} \alpha b^I b^J b^K \right) (dt du \sigma_1 \sigma_2 \tilde{\sigma}_3) . \quad (\text{A.6})$$

A naive evaluation of the near horizon action would fail to include the second contribution (A.5). An alternative way to arrive at the correct answer (A.6) is to take a dimensional reduction along the σ_3 direction to make the problem effectively four dimensional [37].

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